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Braid-group approach to the derivation of universal $\check{R}$ matrices

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Abstract. A new method for deriving universal $\check{R}$ matrices from braid-group representation is discussed. In this case, universal $\check{R}$ operators can be defined and expressed in terms of products of braid-group generators. The advantage of this method is that matrix elements of $\check{R}$ are rank independent, and leaves multiplicity problem-concerning coproducts of the corresponding quantum groups untouched. As examples, $\check{R}$-matrix elements of $[1]\times[1]$, $[2]\times[2]$, $[1^2]\times[1^2]$, and $[21]\times[21]$ with multiplicity two for A_n-type and $[1]\times[1]$ for B_n-type, C_n-type, and D_n-type quantum groups, which are related to Hecke algebra and Birman–Wenzl algebra, respectively, are derived by using this method.

1. Introduction

Universal $\check{R}$ matrices are solutions of spectral parameter-free Yang–Baxter equations (YBEs). YBE are of importance in both mathematics and physics, such as statistical models [1], scattering matrices [2], knot theory [3], conformal field theory (CFT) [4] and so on. Once the parameter-free $\check{R}$ matrices are known, the parameter-dependent matrix $\check{\boldsymbol{R}}(x)$ can be obtained by using the so-called Baxterization procedure [5–7]. Up to now the derivations of the standard $\check{R}$ matrices have been obtained through the representation theory of quantum groups by many authors including Drinfeld [8], Jimbo [9, 10] and Reshetikhin [11], and by taking limit of statistical models [12–14] or by using Witten's approach of the link polynomials [15, 16]. There are also many other methods to construct $\check{R}$ matrices [17–19]. Based on these methods, various classes of $\check{R}$ matrices have been obtained, which can easily be found in the current mathematical-physics literature. From these methods it can easily be seen that $\check{R}$ matrices are related either to the tensor products of generators or to the Clebsch–Gordan (CG) coefficients of the corresponding quantum groups. Thus knowledge of representation theory of quantum groups, such as coupling coefficients, projection operators, and so on are very important in these methods to construct the standard $\check{R}$ matrices. In some cases, the multiplicity problem will be involved in the coproducts of the corresponding quantum groups, which is very complicated to solve. Secondly, the $\check{R}$ matrix, usually expressed in terms of the CG matrix with summing over all the possible resultant irreps of the corresponding quantum groups, can only be derived for specific n, e.g. of A_n, or B_n at a time. When the n increases, the CG matrix will become very large. It will soon become intractable for higher n due to the drastic increase of the number of the

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CG coefficients involved. It is also well known that matrix representations of braid-group generators can be constructed by using the $\check{R}$ matrix via

$$g_i = 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \check{R} \otimes 1 \otimes \cdots \otimes 1 \tag{1}$$

where $\check{R}$ is in the ith and $(i+1)$th spaces. It can easily be proved that the representations of the braid groups constructed in this way are not irreducible in general. However, on the other hand the $\check{R}$ operator can be regarded as the deformed permutation operator which permutes two representations of the corresponding quantum groups. From this point of view, the $\check{R}$ matrix is representative of $\check{R}$ operators in the uncoupled basis of the corresponding quantum groups.

In this paper, we will outline a new procedure for deriving standard solution of the universal $\check{R}$ matrices from representations of braid groups directly. The $\check{R}$ operators will be expressed in terms of products of the corresponding braid-group generators, which are acting on the vector-product space of the quantum groups. In this case, we only need CG coefficients of $[\lambda]\times[1]$ of the quantum groups. Such CG coefficients are very simple, and of course always multiplicity free. Furthermore, the calculation is rank independent. It opens up a new way to compute $\check{R}$ matrix elements for arbitrary n once and for all, instead of one n at a time. As examples, $\check{R}$ matrix elements of $[1]\times[1]$, $[2]\times[2]$, $[1^2]\times[1^2]$, and $[21]\times[21]$ with multiplicity two for A_n-type, and $[1]\times[1]$ for B_n-, C_n-, and D_n-type quantum groups, which are related to Hecke algebra and Birman–Wenzl algebra, respectively, are derived by using this method.

2. The $\check{R}$ operators

Let $V^{[\lambda_1]}$ and $V^{[\lambda_2]}$ be spaces spanned by basis vectors of irreducible representations of $[\lambda_1]$ and $[\lambda_2]$ of any quantum group. Then, the action of $\check{R}$ is defined by

$$\check{R}(V^{[\lambda_1]} \otimes V^{[\lambda_2]}) \rightarrow V^{[\lambda_2]} \otimes V^{[\lambda_1]}. \tag{2}$$

We assume that maximum rank of $[\lambda_1]$ and $[\lambda_2]$ is f. That is $[\lambda_1]$ and $[\lambda_2]$ can be constructed by, at most, f-fold coproducts of rank-1 tensor operators of the corresponding quantum group. For example, if $[\lambda_1]$ is of maximum rank, we can write

$$T^1 \otimes T^2 \otimes \cdots \otimes T^f \rightarrow T^{[\lambda_1]} \tag{3}$$

where T^i with $i = 1, 2, \ldots, f$ are vector operators in the ith space. In the A-type quantum group case, in order to label the basis of $T^{[\lambda_1]}$, one can assign a Weyl tableau $w^{[\lambda_1]}(\omega_1^0)$ to $T^{[\lambda_1]}$, where $(\omega_1^0) = (1, 2, \ldots, f)$ is used to indicate that the f vectors are coupled to $[\lambda_1]$. Now we assume that the rank of $[\lambda_2^0](\omega_2^0, 2f-k+1, 2f-k+2, \ldots, 2f)$, where $(\omega_2^0) = (f+1, f+2, \ldots, 2f-k)$, while the indices $(2f-k+1, 2f-k+2, \ldots, 2f)$ are used to label the remaining scalars, i.e.

$$T^1 \otimes T^2 \otimes \cdots \otimes T^{2f-k} \otimes 1^{2f-k+1} \otimes \cdots \otimes 1^{2f} \rightarrow T^{[\lambda_2]}. \tag{4}$$

Hence, the uncoupled basis vectors of $|w^{[\lambda_1]}, w^{[\lambda_2]}\rangle$ can be written explicitly as

$$|w^{[\lambda_1]}(\omega_1^0), w^{[\lambda_2]}(\omega_2^0, 2f-k+1, 2f-k+2, \ldots, 2f)\rangle \tag{5}$$

according to the braid-group action, i.e. equation (5) can now be understood as uncoupled basis vectors of braid group B_{2f} under the $\check{R}$ operation with

$$\begin{aligned}\check{R}\{&(\omega_1^0), (\omega_2^0, 2f-k+1, 2f-k+2, \ldots, 2f)\}\\ &\rightarrow \{(\omega_2^0), 2f-k+1, 2f-k+2, \ldots, 2f), (\omega_1)\}.\end{aligned}$$

Under this labelling scheme, we find the $\check{R}$ operator can be expressed in terms of braid-group B_{2f} generators.

$$\check{R}_{f=1} = g_1 \tag{6}$$

$$\check{R}_{f=2} = g_2 g_1 g_3 g_2 = g_2 \check{R}_1 g_3 g_2. \tag{7}$$

Through induction we finally obtain

$$\check{R}_f = g_f g_{f+1} g_{f+2} \ldots g_{2f-2} \check{R}_{f-1} g_{2f-1} g_{2f-2} \ldots g_{f+2} g_{f+1} g_f. \tag{8}$$

Equation (8) gives the universal $\check{R}$ operator for fixed f in braid-group generator product form, where f is the maximum rank of irreps of the corresponding quantum groups. The universality means that the operator given in (8) satisfies (2) for any irrep of any type of the corresponding quantum groups, which is just what Jimbo and Drinfeld referred to [8–10]. The differences between this work and those of Jimbo and Drinfeld are

(i) the universal $\check{R}$ operator is now written in terms of braid-group generator product form

(ii) the $\check{R}$ operator defined in (8) is rank-f dependent.

It seems that the $\check{R}$ operator given in (8) loses some universality. However, one can use it to compute all universal $\check{R}$ matrices because (8) is valid for arbitrary rank f of irreps in the tensor product space of the corresponding quantum groups given in (2). While in Jimbo and Drinfeld's works the $\check{R}$ operator is written in terms of quantum double basis of Hopf algebra, which is rank f independent. Thus (8) can be regarded as a braid-group form of the universal $\check{R}$ operators formerly defined by Jimbo and Drinfeld.

The problem concerning the braid-group realization for the fixed type of the corresponding quantum group has been studied in many works [9, 10, 20], from which one knows that the braid-group realization is Hecke algebra for the A-type quantum groups, is Birman–Wenzl algebra for the B, C and D types, and is Kalfagianni algebra [22] for G_2. However, the problem still remains open for F_4-, and the E-type quantum groups. In the next section, we will outline a procedure for evaluating the $\check{R}$ matrices concerning A-type quantum groups, and will also give a simple example for the B-, C-, and D-type cases.

3. Evaluation of $\check{R}$ matrices

In this section, we will outline a procedure for evaluating $\check{R}$ matrices. We will consider Hecke algebra for the A-type quantum groups and give an example of Birman–Wenzl algebra for the B-, C-, and D-type cases separately.

The $\check{R}$ operator is a braid-group element, which can be expressed in terms of braid-group generators by using (8). $\check{R}$ operates among the coordinate indices $\{1, 2, \ldots, 2f\}$. Any uncoupled basis vectors $|w^{[\lambda_1]}(\omega_1^0), w^{[\lambda_2]}(\omega_2^0, 2f-k+1, 2f-k+2, \ldots, 2f)\rangle$ of any quantum group can further be expanded in terms of uncoupled basis vectors of $2f-k$-fold basic representations, namely

$$|w^{[\lambda_1]}(\omega_1^0), w^{[\lambda_2]}(\omega_2^0, 2f-k+1, 2f-k+2, \ldots, 2f)\rangle = \sum_{\omega} a_\omega Q_\omega |a_1, a_2, \ldots, a_{2f-k}, 1^{2k-k+1}, 1^{2f-k+2}, \ldots, 1^{2f}\rangle \tag{9}$$

where a_ω can be obtained by using the CG coefficients for the coupling $((1\otimes)^f)^{[\lambda_1]}((1\otimes)^{f-2k})^{[\lambda_2]}$ of the corresponding quantum group, $\{a_1, a_2, \ldots, a_{2f-k}\}$ are the vector components of the quantum group satisfying the normal ordering $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_{2f-k}$, and Q_ω is the left coset representative in the decomposition

$$B_{2f} = \sum_{\omega} \oplus Q_\omega (B_1 \times B_1 \times \cdots \times B_1). \tag{10}$$

For example, using the CG coefficients of $U_q(2)$ tabulated in [21], we have

$$|aa, ab\rangle = \sqrt{\frac{q^{-1}}{[2]}}|a, a, a, b\rangle + \sqrt{\frac{q}{[2]}}|a, a, b, a\rangle = \left(\sqrt{\frac{q^{-1}}{[2]}} + \sqrt{\frac{q}{[2]}}g_3\right)|a, a, a, b\rangle \tag{11a}$$

$$\begin{aligned}|ab, aa\rangle &= \sqrt{\frac{q^{-1}}{[2]}}|a, b, a, a\rangle + \sqrt{\frac{q}{[2]}}|b, a, a, a\rangle \\ &= \left(\sqrt{\frac{q^{-1}}{[2]}}g_2g_3 + \sqrt{\frac{q}{[2]}}g_1g_2g_3\right)|a, a, a, b\rangle\end{aligned} \tag{11b}$$

where $[x]$ is the q-number of x. The vector-space indices are arranged in natural order, e.g.

$$|a, b, c\rangle \sim T_a^1 T_b^2 T_c^3 \tag{12}$$

and the uncoupled basis vectors $Q_\omega|a_1, a_2, \ldots, a_{2f-k}, 1^{2f-k+1}, 1^{2f-k+2}, \ldots, 1^{2f}\rangle$ with different ω and a_is are orthonormal:

$$\begin{aligned}&\langle a_1', a_2', \ldots, a_{2f-k}', 1^{2f-k+1}, 1^{2f-k+2}, \ldots, 1^{2f}|Q_{\omega'}^\dagger Q_\omega|a_1, a_2, \ldots \\ &\quad \ldots a_{2f-k}, 1^{2f-k+1}, 1^{2f-k+2}, \ldots, 1^{2f}\rangle = \delta_{\omega'\omega}\prod \delta_{a_i'a_i}.\end{aligned} \tag{13}$$

That is, we use the orthogonal uncoupled basis of the quantum group. In this case, it can be proved that the braid-group parameters, e.g. q, should be real, otherwise (13) will no longer be valid. Equivalently, we have used the following star operation

$$g_i^\dagger = g_i \qquad \text{for } i = 1, 2, \ldots, 2f-1. \tag{14}$$

However, results for generic braid-group parameters can be obtained through analytical continuation, i.e. the final results are valid for generic parameters as well.

The action of g_i on the basis vectors $|a_1, a_2, \ldots, a_{2f-k}, 1^{2f-k+1}, 1^{2f-k+2}, \ldots, 1^{2f}\rangle$ is given by the following rule

$$\begin{aligned}&g_i|a_1, a_2, \ldots, a_{2f-k}, 1^{2f-k+1}, 1^{2f-k+2}, \ldots, 1^{2f}\rangle \\ &\quad = q|a_1, a_2, \ldots, a_{2f-k}, 1^{2f-k+1}, 1^{2f-k+2}, \ldots, 1^{2f}\rangle\end{aligned} \tag{15a}$$

if the components a_i and a_{i+1} are the same. This rule can be proved by using the symmetrization method outlined in [21]. While

$$\begin{aligned}&g_i|a_1, a_2, \ldots a_{2f-k}, 1^{2f-k+1}, 1^{2f-k+2}, \ldots, 1^{2f}\rangle \\ &\quad = |a_1, \ldots, a_{i+1}, a_i, \ldots, a_{2f-k}, 1^{2f-k+1}, 1^{2f-k+2}, \ldots, 1^{2f}\rangle\end{aligned} \tag{15b}$$

if the components a_i and a_{i+1} are different. It should be noted that because of the property of braid groups, we should always write the uncoupled basis vectors in the operator form. For example, in the Hecke-algebra case

$$g_1|a, b\rangle = |b, a\rangle \tag{16a}$$

$$g_1|b, a\rangle = g_i^2|a, b\rangle = (q - q^{-1})|b, a\rangle + |a, b\rangle \tag{16b}$$

otherwise the notation $|b, a\rangle$ is rather confusing in the practical computation.

Hence, the action of the operator $\check{R}$ on the uncoupled basis vectors of quantum groups is well defined. Using (15) and defining relations among braid-group generators, we can derive the $\check{R}$-matrix elements. In the following, we will outline how to use this procedure to derive $\check{R}$-matrix elements. First, we will discuss the Hecke-algebra case. Then, we will show a simple example in the Birman–Wenzl-algebra case.

3.1. Hecke-algebra case

The Hecke algebra $H_f(q)$ is generated by $f-1$ elements $g_1, g_2, \ldots, g_{f-1}$, which satisfy the following well known braid relations

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \tag{17a}$$

$$g_i g_j = g_j g_i \qquad \text{for } |i-j| \geqslant 2 \tag{17b}$$

$$(g_i)^2 = g_i(q-q^{-1}) + 1. \tag{17c}$$

The braid-group elements $\check{R}$, which can be expressed in terms of braid-group generators by using (8), are operating among the vector space indices $\{1, 2, \ldots, 2f\}$. Any uncoupled irreducible basis vectors $|w^{[\lambda_1]}(\omega_1^0), w^{[\lambda_2]}(\omega_2^0, 2f-k+1, 2f-k+2, \ldots, 2f)\rangle$ of $U_q(n)$ can further be expanded in terms of uncoupled basis vectors of $2f-k$-fold basic representations are given by (9). In the following, we restrict ourselves to $[\lambda_1] = [\lambda_2]$, and use $[2] \times [2]$ as an example.

Step 1. Write out all the uncoupled basis vectors of the corresponding quantum group.

In the $[2] \times [2] U_q(n)$ case, these are

$$|ii, ii\rangle = |i, i, i, i\rangle \tag{18a}$$

$$\begin{aligned} |ij, ij\rangle &= \left(\sqrt{\frac{q^{-1}}{[2]}}|i, j, 1, 1\rangle + \sqrt{\frac{q}{[2]}}|j, i, 1, 1\rangle\right)\left(\sqrt{\frac{q^{-1}}{[2]}}|1, 1, i, j\rangle + \sqrt{\frac{q}{[2]}}|1, 1, j, i\rangle\right) \\ &= A_{12} A_{34} g_2 |i, i, j, j\rangle \qquad \text{for } i < j \end{aligned} \tag{18b}$$

where

$$A_{12} = \sqrt{\frac{1}{[2]}}(q^{-1/2} + q^{1/2} g_1) \tag{19a}$$

$$A_{34} = \sqrt{\frac{1}{[2]}}(q^{-1/2} + q^{1/2} g_3) \tag{19b}$$

where the CG coefficients of $[1] \times [1] \downarrow [2]$ of $U_q(n)$ have been used. Similarly, we have

$$|ij, kl\rangle = A_{12} A_{34} |i, j, k, l\rangle \qquad \text{for } i < j < k < l \tag{19c}$$

while

$$|kl, ij\rangle = \check{R}|ij, kl\rangle = g_2 g_3 g_1 g_2 |ij, kl\rangle. \tag{19d}$$

All the other basis vectors can thus be written out similarly.

Step 2. Derive algebraic relations among $\check{R}$, A_{12}, A_{34}.

It can be proved that

$$\check{R} A_{12} = A_{34} \check{R} \tag{20a}$$

$$\check{R} A_{34} = A_{12} \check{R}. \tag{20b}$$

Thus, we obtain

$$\check{R} A_{12} A_{34} = A_{12} A_{34} \check{R}. \tag{21}$$

Equation (21) is very useful in the practical computation. We also need the following quadratic equation of $\check{R}$:

$$\begin{aligned} \check{R}^2 = {} & (q-q^{-1}) g_1 g_3 \check{R} + (q-q^{-1})^2 \check{R} + (q-q^{-1}) g_3 g_1 g_2 g_1 g_3 + (q-q^{-1}) g_2 g_1 g_2 \\ & + (q-q^{-1}) g_2 g_3 g_2 + (q-q^{-1}) g_2 + 1. \end{aligned} \tag{22}$$

Step 3. Applying $\check{R}$ on all uncoupled basis vectors obtained in step 1 and using the algebraic relations derived in step 2 and (15), we thus obtain all the $\check{R}$-matrix elements in this step. For example, in [2] × [2] of $U_q(n)$ case, we have

$$\check{R}|ii, ii\rangle = g_2 g_1 g_3 g_2 |i, i, i, i\rangle = q^4 |ii, ii\rangle \tag{23}$$

$$\begin{aligned} \check{R}|ij, |ij\rangle &= \check{R} A_{12} A_{34} |i, j, i, j\rangle = \check{R} A_{12} A_{34} g_2 |i, i, j, j\rangle = A_{12} A_{34} g_2 g_3 g_1 g_2^2 |i, i, j, j\rangle \\ &= A_{12} A_{34} g_2 g_3 g_1 g_2 (q - q^{-1}) |i, i, j, j\rangle + A_{12} A_{34} g_2 g_3 g_1 |i, i, j, j\rangle \\ &= A_{12} A_{34} (q - q^{-1}) |j, j, i, i\rangle + q^2 A_{12} A_{34} |i, j, i, j\rangle. \end{aligned} \tag{24}$$

Using the following relation

$$A_{12}|i, i\rangle = q^{1/2}[2]^{1/2}|i, i\rangle \tag{25}$$

we obtain

$$\check{R}|ij, ij\rangle = (q^3 - q^{-1})|jj, ii\rangle + q^2|ij, ij\rangle \qquad \text{for } i < j. \tag{26}$$

Similarly, we have

$$\check{R}|ij, kl\rangle = |kl, ij\rangle \qquad \text{for } i < j < k < l \tag{27a}$$

$$\begin{aligned} \check{R}|kl, ij\rangle = \check{R}^2|ij, kl\rangle &= (q - q^{-1})^2(q^2 + 1)|kl, ij\rangle + (q^3 - q^{-1})|jl, ik\rangle + (q^2 - 1)|jk, il\rangle \\ &+ (q^2 - 1)|il, jk\rangle + (q - q^{-1})|ik, jl\rangle + |ij, lk\rangle \qquad \text{for } i < j < k < l. \end{aligned} \tag{27b}$$

All the other $\check{R}$-matrix elements can thus be derived by using this method. In the next section we list all the $\check{R}$-matrix elements for [1] × [1], [2] × [2], $[1^2] \times [1^2]$, and some of [21] × [21] for $U_q(n)$.

3.2. *A simple example of the Birman–Wenzl-algebra case*

Birman–Wenzl algebra $C_f(r, q)$ is generated by $\{g_i; i = 1, 2, \ldots, f - 1\}$, which satisfy the braid-group relations (17*a* and *b*) with constraint

$$(g_i - r^{-1})(g_i - q)(g_i + q^{-1}) = 0. \tag{28}$$

One can also introduce $f - 1$ auxiliary generators $\{e_i; i = 1, 2, \ldots, f - 1\}$ with

$$e_i = 1 - \frac{g_i - g_i^{-1}}{q - q^{-1}}. \tag{29}$$

The following relations are helpful in practical computation

$$e_i g_i = r^{-1} e_i \tag{30a}$$

$$e_i^2 = x e_i. \tag{30b}$$

Using the above-defined relations, one obtains

$$g_i^2 = (q - q^{-1})(g_i - r^{-1} e_i) + 1. \tag{31}$$

We consider B_n, C_n, and D_n cases with irreps [1] × [1]. In this case $r = q^{2n}$ for B_n, $r = q^{2n-1}$ for D_n, and $r = q^{-2n-1}$ for C_n. The procedure for evaluating the $\check{R}$-matrix elements in this case is similar to that for those of the Hecke-algebra case. First, we write out all the uncoupled basis vectors of [1] × [1]. These are $|\mu, \nu\rangle$ for $-n \leqslant \mu, \nu \leqslant n$. The $\check{R}$ operator is g_1. However, the e_1 operator is so-called q-deformed trace contraction operator defined by

$$e_1|\mu, -\nu\rangle = (-)^{\mu} \delta_{\mu\nu} \sum_k (-)^k q^k |k, -k\rangle \tag{32}$$

when it is applied to the uncoupled basis vectors of the corresponding quantum groups. Using (32), and assuming that the uncoupled basis vectors are orthogonal, one can prove that

$$g_1|\mu, -\mu\rangle = r^{-1}|-\mu, \mu\rangle \qquad \text{for } \mu \neq 0. \tag{33a}$$

If both μ and ν are non-zero, we can use (15) to derive the $\check{R}$-matrix elements. These are

$$g_1|\mu, \mu\rangle = q|\mu, \mu\rangle \tag{33b}$$

$$g_1|\mu, \nu\rangle = |\nu, \mu\rangle \qquad \text{for } \mu > \nu, \mu \neq -\nu. \tag{33c}$$

Then, using (31), we get

$$g_1|\nu, \mu\rangle = (q - q^{-1})|\nu, \mu\rangle + |\mu, \nu\rangle \qquad \text{for } \mu > \nu, \mu \neq -\nu \tag{33d}$$

$$\begin{aligned} g_1|-\mu, \mu\rangle = {} & (q^{\mu-1} - q^{\mu+1} + r)|\mu, -\mu\rangle + (q - q^{-1} - q^{1-\mu} + q^{-\mu-1})|-\mu, \mu\rangle \\ & -(q - q^{-1}) \sum_{k \neq \mu, -\mu} q^k (-)^{\mu+k} |k, -k\rangle. \end{aligned} \tag{33e}$$

In the B_n case, we need uncoupled basis vectors $|0, 0\rangle$. Using (30), and (32), we obtain

$$g_1|0, 0\rangle = \sum_{\mu>0} (-)^\mu a_\mu |-\mu, \mu\rangle + \sum_{\mu>0} (-)^\mu b_\mu |\mu, -\mu\rangle + c_0|0, 0\rangle \tag{33f}$$

where

$$\begin{aligned} a_\mu = {} & q^{-\mu} r^{-1} - q^{-\mu+1} + q^{-\mu-1} + q^{1-2\mu} - q^{-2\mu-1} - q^\mu r^{-1} \\ & + \frac{q - q^{-1}}{q - 1} (q^{-\mu} - q^{-n-\mu} - q^{-2\mu+1} + q^{-2\mu}) \end{aligned} \tag{34a}$$

$$b_\mu = r^{-1} q^\mu - q^{-1} + q - r q^{-\mu} + \frac{q - q^{-1}}{q - 1} (q^\mu - q^{-n+\mu} - q + 1) \tag{34b}$$

$$c_0 = r^{-1} + \frac{q - q^{-n+1} - q^{-1} + q^{-n-1}}{q - 1}. \tag{34c}$$

Using (31), we can prove that the basis vectors $|-\mu, \mu\rangle$, and $|0, 0\rangle$ are not orthonormal. The normalized basis vectors of $[1] \times [1]$ of B, C, and D cases are

$$\begin{aligned} & |\mu, \mu) = |\mu, \mu\rangle \\ & |\mu, \nu) = |\mu, \nu\rangle \qquad \text{for } \mu > \nu, \mu \neq -\nu \\ & |\nu, \mu) = |\nu, \mu\rangle \qquad \text{for } \mu > \nu \\ & |\mu, -\mu) = |\mu, -\mu\rangle \qquad \text{for } \mu > 0 \\ & |-\mu, \mu) = \frac{1}{N_\mu} |-\mu, \mu\rangle \qquad \text{for } \mu > 0 \\ & |0, 0) = \frac{1}{N_0} |0, 0\rangle \end{aligned} \tag{35a}$$

where

$$\begin{aligned} & N_\mu = \sqrt{r^2 - r q^\mu (q - q^{-1})} \\ & N_0^2 = \frac{\sum_{\mu>0} \{a_\mu^2 r (r - q^{\mu+1} + q^{\mu-1}) + b_\mu^2\}}{1 + (q - q^{-1})(c_0 - r^{-1}) + c_0^2}. \end{aligned} \tag{35b}$$

If one knows the CG coefficients for $[\lambda] \times [1]$ for the corresponding quantum groups, one can also derive other $\check{R}$-matrix elements as have been done in the A_n case.

4. Some $\check{R}$-matrix elements of the $U_q(n)$ case

4.1. [1] × [1] irrep

In this case $\check{R} = g_1$. We have

$$\begin{aligned}
&\check{R}|i,i\rangle = q|i,i\rangle \\
&\check{R}|i,j\rangle = |j,i\rangle \\
&\check{R}|j,i\rangle = (q-q^{-1})|j,i\rangle + |i,j\rangle.
\end{aligned} \tag{36}$$

4.2. [2] × [2] irrep

In this case $\check{R} = g_2 g_1 g_3 g_2$. We have

$$\begin{aligned}
&\check{R}|ii,ii\rangle = q^4|ii,ii\rangle \\
&\check{R}|ii,ij\rangle = q^2|ij,ii\rangle \\
&\check{R}|ij,jj\rangle = q^2|jj,ij\rangle \\
&\check{R}|ii,jj\rangle = |jj,ii\rangle \\
&\check{R}|ij,kl\rangle = |kl,ij\rangle \\
&\check{R}|ij,kk\rangle = |kk,ij\rangle \\
&\check{R}|ik,jj\rangle = |jj,ik\rangle + (q-q^{-1})(q[2])^{1/2}|jk,ij\rangle \\
&\check{R}|jj,ik\rangle = (q-q^{-1})(q[2])^{1/2}|jk,ij\rangle + |ik,jj\rangle \\
&\check{R}|ik,jl\rangle = (q-q^{-1})|kl,ij\rangle + |jl,ik\rangle \\
&\check{R}|ij,ij\rangle = (q^3-q^{-1})|jj,ii\rangle + q^2|ij,ij\rangle \\
&\check{R}|ij,ii\rangle = q^2|ii,ij\rangle + (q^4-1)|ij,ii\rangle \\
&\check{R}|jj,ij\rangle = q^2|ij,jj\rangle + (q^4-1)|jj,ij\rangle \\
&\check{R}|il,jk\rangle = |jk,ij\rangle + (q-q^{-1})|jl,ik\rangle + (q^2-1)|kl,ij\rangle \\
&\check{R}|jk,il\rangle = (q-q^{-1})|jl,ik\rangle + |il,jk\rangle + (q^2-1)|kl,ij\rangle \\
&\check{R}|jl,ik\rangle = |ik,jl\rangle + (q-q^{-1})(|jk,il\rangle + il,jk\rangle + (q-q^{-1})^2|jl,ik\rangle + (q^3-q)|kl,ij\rangle \\
&\check{R}|jj,ii\rangle = (q^4-q^2-1+q^{-2})|jj,ii\rangle + |ii,jj\rangle + (q^3-q^{-1})|ij,ij\rangle \\
&\check{R}|jk,ij\rangle = (2q^2+q^{-2}-3)|jk,ij\rangle + q|ij,jk\rangle + (q-q^{-1})(q[2])^{1/2}(|ik,jj\rangle + |jj,ik\rangle \\
&\check{R}|kl,ij\rangle = (q-q^{-1})^2(q^2+1)|kl,ij\rangle + (q^3-q)|jl,ik\rangle + (q^2-1)|jk,il\rangle \\
&\qquad\qquad +(q^2-1)|il,jk\rangle + (q-q^{-1})|ik,jl\rangle + |ij,lk\rangle \\
&\check{R}|ij,ik\rangle = q|ik,ij\rangle + (q^{3/2}-q^{-1/2})[2]|jk,ii\rangle \\
&\check{R}|ik,ij\rangle = q|ij,ik\rangle + (q^2-1)|ik,ij\rangle + (q^2-1)(q[2])^{1/2}|jk,ii\rangle \\
&\check{R}|jk,ii\rangle = (q-q^{-1})^2[2]q|jk,ii\rangle + |ii,jk\rangle + (q^2-1)(q[2])^{1/2}|ik,ij\rangle \\
&\qquad\qquad +(q-q^{-1})|ij,ik\rangle \\
&\check{R}|kk,ij\rangle = q|ij,kk\rangle + q^{3/2}(q-q^{-1})^2[2]^2|kk,ij\rangle + (q^2-1)(q[2])^{1/2}|jk,ik\rangle \\
&\qquad\qquad +(q-q^{-1})(q[2])^{1/2}|ik,jk\rangle \\
&\check{R}|ik,jk\rangle = q|jk,ik\rangle + (q-q^{-1})([2]q)^{1/2}|kk,ij\rangle \\
&\check{R}|jk,ik\rangle = (q^2-1)|jk,ik\rangle + q|ik,jk\rangle + (q^2-1)(q[2])^{1/2}|kk,ij\rangle.
\end{aligned} \tag{37}$$

4.3. $[1^2] \times [1^2]$ irrep

$$
\begin{aligned}
&\check{R}\left|\begin{matrix} i & , & i \\ j & & j\end{matrix}\right\rangle = q^2\left|\begin{matrix} i & , & i \\ j & & j\end{matrix}\right\rangle\\
&\check{R}\left|\begin{matrix} i & , & i \\ j & & k\end{matrix}\right\rangle = q\left|\begin{matrix} i & , & i \\ k & & j\end{matrix}\right\rangle\\
&\check{R}\left|\begin{matrix} i & , & j \\ j & & k\end{matrix}\right\rangle = q\left|\begin{matrix} j & , & i \\ k & & j\end{matrix}\right\rangle\\
&\check{R}\left|\begin{matrix} i & , & j \\ k & & k\end{matrix}\right\rangle = q\left|\begin{matrix} j & , & i \\ k & & k\end{matrix}\right\rangle\\
&\check{R}\left|\begin{matrix} i & , & j \\ k & & l\end{matrix}\right\rangle = \left|\begin{matrix} j & , & i \\ l & & k\end{matrix}\right\rangle + (q-q^{-1})\left|\begin{matrix} k & , & i \\ l & & j\end{matrix}\right\rangle\\
&\check{R}\left|\begin{matrix} i & , & i \\ k & & j\end{matrix}\right\rangle = (q^2-1)\left|\begin{matrix} i & , & i \\ k & & j\end{matrix}\right\rangle + q\left|\begin{matrix} i & , & i \\ j & & k\end{matrix}\right\rangle\\
&\check{R}\left|\begin{matrix} j & , & i \\ k & & j\end{matrix}\right\rangle = q\left|\begin{matrix} i & , & j \\ j & & k\end{matrix}\right\rangle + (q^2-1)\left|\begin{matrix} j & , & i \\ k & & j\end{matrix}\right\rangle\\
&\check{R}\left|\begin{matrix} j & , & i \\ k & & k\end{matrix}\right\rangle = q\left|\begin{matrix} i & , & j \\ k & & k\end{matrix}\right\rangle + (q^2-1)\left|\begin{matrix} j & , & i \\ k & & k\end{matrix}\right\rangle\\
&\check{R}\left|\begin{matrix} k & , & i \\ l & & j\end{matrix}\right\rangle = q^{-1}[2](q-q^{-1})^2\left|\begin{matrix} k & , & i \\ l & & j\end{matrix}\right\rangle + (q^{-1}-q^{-3})\left|\begin{matrix} j & , & i \\ l & & k\end{matrix}\right\rangle\\
&\qquad -(1-q^{-2})\left|\begin{matrix} j & , & i \\ k & & l\end{matrix}\right\rangle - (1-q^{-2})\left|\begin{matrix} i & , & j \\ l & & k\end{matrix}\right\rangle\\
&\qquad +(q-q^{-1})\left|\begin{matrix} i & , & j \\ k & & l\end{matrix}\right\rangle + \left|\begin{matrix} i & , & k \\ j & & l\end{matrix}\right\rangle\\
&\check{R}\left|\begin{matrix} j & , & i \\ l & & k\end{matrix}\right\rangle = (q-q^{-1})^2\left|\begin{matrix} j & , & i \\ l & & k\end{matrix}\right\rangle + (q^{-1}-q^{-3})\left|\begin{matrix} k & , & i \\ l & & j\end{matrix}\right\rangle + (q-q^{-1})\left|\begin{matrix} j & , & i \\ k & & l\end{matrix}\right\rangle\\
&\qquad +(q-q^{-1})\left|\begin{matrix} i & , & j \\ l & & k\end{matrix}\right\rangle + \left|\begin{matrix} i & , & j \\ k & & l\end{matrix}\right\rangle\\
&\check{R}\left|\begin{matrix} i & , & j \\ l & & k\end{matrix}\right\rangle = \left|\begin{matrix} j & , & i \\ k & & l\end{matrix}\right\rangle + (q-q^{-1})\left|\begin{matrix} j & , & i \\ l & & k\end{matrix}\right\rangle - (1-q^{-2})\left|\begin{matrix} k & , & i \\ l & & j\end{matrix}\right\rangle\\
&\check{R}\left|\begin{matrix} j & , & i \\ k & & l\end{matrix}\right\rangle = \left|\begin{matrix} i & , & j \\ l & & k\end{matrix}\right\rangle - (1-q^{-2})\left|\begin{matrix} k & , & i \\ l & & j\end{matrix}\right\rangle + (q-q^{-1})\left|\begin{matrix} j & , & i \\ l & & k\end{matrix}\right\rangle .
\end{aligned}
\tag{38}
$$

4.4. $[21] \times [21]$ irrep

In this case $\check{R} = g_3g_4g_2g_3g_1g_2g_5g_4g_3$. Using the CGCs for $[1] \times [1] \downarrow [2]$ or $[11]$, and $[2] \times [1] \downarrow [21]$ or $[11] \times [1] \downarrow [21]$, we have the following expansions for uncoupled basis vectors of $[21] \times [21]$.

$$
\begin{aligned}
&\left|\begin{matrix} ij \\ j\end{matrix}\right\rangle = \sqrt{\frac{q}{[3]!}}(1+qg_1-q^{-1}[2]g_2g_1)|i,j,j\rangle = B^0_{12}|i,j,j\rangle\\
&\left|\begin{matrix} ii \\ j\end{matrix}\right\rangle = \sqrt{\frac{q}{[3]!}}([2]-q^{-2}g_2-q^{-1}g_1g_2)|i,i,j\rangle = B^1_{12}|i,j,j\rangle\\
&\left|\begin{matrix} ik \\ j\end{matrix}\right\rangle = \frac{1}{[2]}(g_2+qg_1g_2-q^{-1}g_2g_1-g_2g_1g_2)|i,j,k\rangle = B^2_{12}|i,j,k\rangle
\end{aligned}
$$

$$\left|\begin{matrix} ij \\ k \end{matrix}\right\rangle = \sqrt{\frac{1}{[3]}}\left(1 + qg_1 - \frac{q^{-2}}{[2]}g_2 - \frac{q^{-1}}{[2]}g_1g_2 - \frac{q^{-1}}{[2]}g_2g_1 - \frac{1}{[2]}g_1g_2g_1\right)|i, j, k\rangle$$
$$= B_{12}^3|i, j, k\rangle. \tag{39}$$

We can also prove that

$$\check{R}B_{12}^pB_{45}^r = B_{45}^pB_{12}^r\check{R} \qquad \text{for } 0 \leqslant p, r \leqslant 3, \tag{40a}$$

and

$$\check{R}B_{12}^p = B_{45}^p\check{R} \qquad \check{R}B_{45}^p = B_{12}^p\check{R} \qquad \text{for } 0 \leqslant p \leqslant 3. \tag{40b}$$

After a long calculation with the help of Hecke-algebra relations defined by (17), we can also derive the following quadratic equation for $\check{R}$

$$\begin{aligned}\check{R}^2 = {} & (q-q^{-1})^2\check{R}g_1g_2g_5g_4g_3 + (q-q^{-1})^3\check{R} + (q-q^{-1})^2\check{R}g_1g_2g_1g_5g_4g_5g_3 \\ & +(q-q^{-1})^3g_2\check{R}g_2 + (q-q^{-1})^3g_5\check{R}g_4g_5g_1 + (q-q^{-1})^3\check{R}g_1g_4 \\ & +(q-q^{-1})^2(g_2g_3g_4g_5g_4g_1g_2g_1g_3g_2 + g_2g_3g_4g_5g_4g_3g_2g_4g_3g_4 \\ & +g_2g_3g_4g_5g_4g_3g_2g_3 + g_3g_1g_2g_1g_3g_4g_5g_4g_3g_2g_3g_1 \\ & +g_3g_4g_5g_4g_1g_3g_2g_3g_4g_3g_5g_1 + g_3g_4g_3g_5g_1g_2g_1g_4g_3g_4 \\ & +g_3g_4g_1g_2g_3g_1g_2g_4g_3g_1 + g_3g_4g_2g_1g_5g_2g_3 + g_3g_4g_2g_3g_5g_2g_4g_3 \\ & +g_3g_4g_2g_5g_4g_3 + g_3g_4g_2g_3g_1g_2g_4g_3 + g_3g_2g_4g_3 + g_5g_3g_4g_1g_2g_3g_4g_5g_2g_1 \\ & +g_4g_3g_2g_4g_3g_4) + (q-q^{-1})(g_2g_3g_4g_5g_4g_3g_2 + g_3g_4g_1g_2g_3g_2g_4g_3g_1 \\ & +g_3g_4g_5g_4g_3 + g_3g_4g_1g_2g_1g_4g_3 + g_3g_4g_3 + g_3 + g_3g_2g_3 \\ & +g_5g_4g_1g_2g_3g_4g_5g_2g_1 + g_3g_4g_2g_3g_4) + 1. \end{aligned} \tag{41}$$

Using these relations, we get

$$\check{R}\left|\begin{matrix} ii \\ j \end{matrix}\;,\;\begin{matrix} ii \\ j \end{matrix}\right\rangle = q^5\left|\begin{matrix} ii \\ j \end{matrix}\;,\;\begin{matrix} ii \\ j \end{matrix}\right\rangle$$
$$\check{R}\left|\begin{matrix} ij \\ j \end{matrix}\;,\;\begin{matrix} ij \\ j \end{matrix}\right\rangle = q^5\left|\begin{matrix} ij \\ j \end{matrix}\;,\;\begin{matrix} ij \\ j \end{matrix}\right\rangle$$
$$\check{R}\left|\begin{matrix} ii \\ j \end{matrix}\;,\;\begin{matrix} ij \\ j \end{matrix}\right\rangle = q^4\left|\begin{matrix} ij \\ j \end{matrix}\;,\;\begin{matrix} ii \\ j \end{matrix}\right\rangle$$
$$\check{R}\left|\begin{matrix} ik \\ k \end{matrix}\;,\;\begin{matrix} jk \\ k \end{matrix}\right\rangle = q^4\left|\begin{matrix} jk \\ k \end{matrix}\;,\;\begin{matrix} ik \\ k \end{matrix}\right\rangle$$
$$\check{R}\left|\begin{matrix} ij \\ j \end{matrix}\;,\;\begin{matrix} jj \\ k \end{matrix}\right\rangle = q^4\left|\begin{matrix} jj \\ k \end{matrix}\;,\;\begin{matrix} ij \\ j \end{matrix}\right\rangle$$
$$\check{R}\left|\begin{matrix} ii \\ j \end{matrix}\;,\;\begin{matrix} ii \\ k \end{matrix}\right\rangle = q^4\left|\begin{matrix} ii \\ k \end{matrix}\;,\;\begin{matrix} ii \\ j \end{matrix}\right\rangle$$
$$\check{R}\left|\begin{matrix} ij \\ j \end{matrix}\;,\;\begin{matrix} ik \\ j \end{matrix}\right\rangle = q^3\left|\begin{matrix} ik \\ j \end{matrix}\;,\;\begin{matrix} ij \\ j \end{matrix}\right\rangle$$
$$\check{R}\left|\begin{matrix} ik \\ j \end{matrix}\;,\;\begin{matrix} ik \\ k \end{matrix}\right\rangle = q^3\left|\begin{matrix} ik \\ k \end{matrix}\;,\;\begin{matrix} ik \\ j \end{matrix}\right\rangle$$
$$\check{R}\left|\begin{matrix} ii \\ j \end{matrix}\;,\;\begin{matrix} ij \\ k \end{matrix}\right\rangle = q^3\left|\begin{matrix} ij \\ k \end{matrix}\;,\;\begin{matrix} ii \\ j \end{matrix}\right\rangle$$
$$\check{R}\left|\begin{matrix} ij \\ k \end{matrix}\;,\;\begin{matrix} jk \\ k \end{matrix}\right\rangle = q^3\left|\begin{matrix} jk \\ k \end{matrix}\;,\;\begin{matrix} ij \\ k \end{matrix}\right\rangle$$
$$\check{R}\left|\begin{matrix} ik \\ j \end{matrix}\;,\;\begin{matrix} ik \\ j \end{matrix}\right\rangle = q^3\left|\begin{matrix} ik \\ j \end{matrix}\;,\;\begin{matrix} ii \\ j \end{matrix}\right\rangle$$

$$\check{R}\left|\begin{matrix} ik \\ j \end{matrix}\;,\;\begin{matrix} jk \\ k \end{matrix}\right\rangle = q^3\left|\begin{matrix} jk \\ k \end{matrix}\;,\;\begin{matrix} ik \\ j \end{matrix}\right\rangle$$

$$\check{R}\left|\begin{matrix} ii \\ j \end{matrix}\;,\;\begin{matrix} jj \\ k \end{matrix}\right\rangle = q^2\left|\begin{matrix} jj \\ k \end{matrix}\;,\;\begin{matrix} ii \\ j \end{matrix}\right\rangle$$

$$\check{R}\left|\begin{matrix} ii \\ k \end{matrix}\;,\;\begin{matrix} jk \\ k \end{matrix}\right\rangle = q^2\left|\begin{matrix} jk \\ k \end{matrix}\;,\;\begin{matrix} ii \\ k \end{matrix}\right\rangle$$

$$\check{R}\left|\begin{matrix} ii \\ j \end{matrix}\;,\;\begin{matrix} ik \\ k \end{matrix}\right\rangle = q^2\left|\begin{matrix} ik \\ k \end{matrix}\;,\;\begin{matrix} ii \\ j \end{matrix}\right\rangle$$

$$\check{R}\left|\begin{matrix} ij \\ j \end{matrix}\;,\;\begin{matrix} jk \\ k \end{matrix}\right\rangle = q^2\left|\begin{matrix} jk \\ k \end{matrix}\;,\;\begin{matrix} ij \\ j \end{matrix}\right\rangle$$

$$\check{R}\left|\begin{matrix} ii \\ j \end{matrix}\;,\;\begin{matrix} jk \\ k \end{matrix}\right\rangle = q\left|\begin{matrix} jk \\ k \end{matrix}\;,\;\begin{matrix} ii \\ j \end{matrix}\right\rangle$$

$$\check{R}\left|\begin{matrix} ij \\ j \end{matrix}\;,\;\begin{matrix} ik \\ k \end{matrix}\right\rangle = q\left|\begin{matrix} ik \\ k \end{matrix}\;,\;\begin{matrix} ij \\ j \end{matrix}\right\rangle + (q^2-1)\left|\begin{matrix} jk \\ k \end{matrix}\;,\;\begin{matrix} ii \\ j \end{matrix}\right\rangle$$

$$\check{R}\left|\begin{matrix} ii \\ k \end{matrix}\;,\;\begin{matrix} jj \\ k \end{matrix}\right\rangle = q\left|\begin{matrix} jj \\ k \end{matrix}\;,\;\begin{matrix} ii \\ k \end{matrix}\right\rangle + (q^2-1)\left|\begin{matrix} jk \\ k \end{matrix}\;,\;\begin{matrix} ii \\ j \end{matrix}\right\rangle$$

$$\check{R}\left|\begin{matrix} ij \\ k \end{matrix}\;,\;\begin{matrix} jj \\ k \end{matrix}\right\rangle = q^3\left|\begin{matrix} jj \\ k \end{matrix}\;,\;\begin{matrix} ij \\ k \end{matrix}\right\rangle + (q^{7/2}-q^{3/2})[2]^{-1/2}\left|\begin{matrix} jk \\ k \end{matrix}\;,\;\begin{matrix} ij \\ j \end{matrix}\right\rangle$$

$$\check{R}\left|\begin{matrix} ii \\ k \end{matrix}\;,\;\begin{matrix} ij \\ k \end{matrix}\right\rangle = q^3\left|\begin{matrix} ij \\ k \end{matrix}\;,\;\begin{matrix} ii \\ k \end{matrix}\right\rangle + (q^{7/2}-q^{3/2})[2]^{-1/2}\left|\begin{matrix} ik \\ k \end{matrix}\;,\;\begin{matrix} ii \\ j \end{matrix}\right\rangle$$

$$\check{R}\left|\begin{matrix} ii \\ k \end{matrix}\;,\;\begin{matrix} ik \\ j \end{matrix}\right\rangle = q^3\left|\begin{matrix} ik \\ j \end{matrix}\;,\;\begin{matrix} ii \\ k \end{matrix}\right\rangle + (q^{7/2}-q^{3/2})\left(\frac{[3]}{[2]}\right)^{1/2}\left|\begin{matrix} ik \\ k \end{matrix}\;,\;\begin{matrix} ii \\ j \end{matrix}\right\rangle$$

$$\check{R}\left|\begin{matrix} ij \\ k \end{matrix}\;,\;\begin{matrix} jj \\ k \end{matrix}\right\rangle = q^3\left|\begin{matrix} jj \\ k \end{matrix}\;,\;\begin{matrix} ik \\ j \end{matrix}\right\rangle + (q^{7/2}-q^{3/2})\left(\frac{[3]}{[2]}\right)^{1/2}\left|\begin{matrix} jk \\ k \end{matrix}\;,\;\begin{matrix} ij \\ j \end{matrix}\right\rangle$$

$$\check{R}\left|\begin{matrix} ij \\ j \end{matrix}\;,\;\begin{matrix} ii \\ j \end{matrix}\right\rangle = (q^5-q^3)\left|\begin{matrix} ij \\ j \end{matrix}\;,\;\begin{matrix} ii \\ j \end{matrix}\right\rangle + q^4\left|\begin{matrix} ii \\ j \end{matrix}\;,\;\begin{matrix} ij \\ j \end{matrix}\right\rangle$$

$$\check{R}\left|\begin{matrix} ii \\ k \end{matrix}\;,\;\begin{matrix} ii \\ j \end{matrix}\right\rangle = (q^5-q^3)\left|\begin{matrix} ii \\ k \end{matrix}\;,\;\begin{matrix} ii \\ j \end{matrix}\right\rangle + q^4\left|\begin{matrix} ii \\ j \end{matrix}\;,\;\begin{matrix} ii \\ k \end{matrix}\right\rangle$$

$$\check{R}\left|\begin{matrix} jj \\ k \end{matrix}\;,\;\begin{matrix} ij \\ j \end{matrix}\right\rangle = (q^5-q^3)\left|\begin{matrix} jj \\ k \end{matrix}\;,\;\begin{matrix} ij \\ j \end{matrix}\right\rangle + q^4\left|\begin{matrix} ij \\ j \end{matrix}\;,\;\begin{matrix} jj \\ k \end{matrix}\right\rangle$$

$$\check{R}\left|\begin{matrix} jk \\ k \end{matrix}\;,\;\begin{matrix} ik \\ k \end{matrix}\right\rangle = q^4\left|\begin{matrix} ik \\ k \end{matrix}\;,\;\begin{matrix} jk \\ k \end{matrix}\right\rangle + (q^5-q^3)\left|\begin{matrix} jk \\ k \end{matrix}\;,\;\begin{matrix} ik \\ k \end{matrix}\right\rangle$$

$$\check{R}\left|\begin{matrix} ij \\ j \end{matrix}\;,\;\begin{matrix} ij \\ k \end{matrix}\right\rangle = q^3\left|\begin{matrix} ij \\ k \end{matrix}\;,\;\begin{matrix} ij \\ j \end{matrix}\right\rangle + (q^{7/2}-q^{3/2})[2]^{1/2}\left|\begin{matrix} jj \\ k \end{matrix}\;,\;\begin{matrix} ii \\ j \end{matrix}\right\rangle$$

$$\check{R}\left|\begin{matrix} ij \\ k \end{matrix}\;,\;\begin{matrix} ik \\ k \end{matrix}\right\rangle = q^3\left|\begin{matrix} ik \\ k \end{matrix}\;,\;\begin{matrix} ij \\ k \end{matrix}\right\rangle + (q^{7/2}-q^{3/2})[2]^{1/2}\left|\begin{matrix} jk \\ k \end{matrix}\;,\;\begin{matrix} ii \\ k \end{matrix}\right\rangle$$

$$\check{R}\left|\begin{matrix} ij \\ k \end{matrix}\;,\;\begin{matrix} ij \\ j \end{matrix}\right\rangle = q^3\left|\begin{matrix} ij \\ j \end{matrix}\;,\;\begin{matrix} ij \\ k \end{matrix}\right\rangle + (q^4-q^2)[2]^{-1}\left|\begin{matrix} ij \\ k \end{matrix}\;,\;\begin{matrix} ij \\ j \end{matrix}\right\rangle$$
$$+(q^4-q^2)\frac{[3]^{1/2}}{[2]}\left|\begin{matrix} ik \\ j \end{matrix}\;,\;\begin{matrix} ij \\ j \end{matrix}\right\rangle + (q^{9/2}-q^{5/2})[2]^{-1/2}\left|\begin{matrix} jj \\ k \end{matrix}\;,\;\begin{matrix} ii \\ j \end{matrix}\right\rangle$$

$$\begin{aligned}\check{R}\left|\begin{array}{lcl} ik & , & ij \\ k & & k \end{array}\right\rangle = {} & q^3\left|\begin{array}{lcl} ij & , & ik \\ k & & k \end{array}\right\rangle + (q^4-q^2)[2]^{-1}\left|\begin{array}{lcl} ik & , & ij \\ k & & k \end{array}\right\rangle \\ & +(q^4-q^2)\frac{[3]^{1/2}}{[2]}\left|\begin{array}{lcl} ik & , & ik \\ k & & j \end{array}\right\rangle + (q^{9/2}-q^{5/2})[2]^{-1/2}\left|\begin{array}{lcl} jk & , & ii \\ k & & k \end{array}\right\rangle\end{aligned}$$

$$\check{R}\left|\begin{array}{lcl} ij & , & ii \\ j & & k \end{array}\right\rangle = q^2\left|\begin{array}{lcl} ii & , & ij \\ k & & j \end{array}\right\rangle + (q^{7/2}-q^{3/2})[2]^{1/2}\left|\begin{array}{lcl} ij & , & ii \\ k & & j \end{array}\right\rangle$$

$$\check{R}\left|\begin{array}{lcl} jj & , & ik \\ k & & k \end{array}\right\rangle = q^2\left|\begin{array}{lcl} ik & , & jj \\ k & & k \end{array}\right\rangle + (q^{7/2}-q^{3/2})[2]^{1/2}\left|\begin{array}{lcl} jk & , & ij \\ k & & k \end{array}\right\rangle$$

$$\begin{aligned}\check{R}\left|\begin{array}{lcl} ii & , & ij \\ k & & j \end{array}\right\rangle = {} & q^2\left|\begin{array}{lcl} ij & , & ii \\ j & & k \end{array}\right\rangle + (q^{7/2}-q^{3/2})[2]^{-1/2}\left|\begin{array}{lcl} ij & , & ii \\ k & & j \end{array}\right\rangle \\ & +(q^{7/2}-q^{3/2})\left(\frac{[3]}{[2]}\right)^{1/2}\left|\begin{array}{lcl} ik & , & ii \\ j & & j \end{array}\right\rangle\end{aligned}$$

$$\begin{aligned}\check{R}\left|\begin{array}{lcl} ik & , & jj \\ k & & k \end{array}\right\rangle = {} & q^2\left|\begin{array}{lcl} jj & , & ik \\ k & & k \end{array}\right\rangle + (q^{7/2}-q^{3/2})[2]^{-1/2}\left|\begin{array}{lcl} jk & , & ij \\ k & & k \end{array}\right\rangle \\ & +(q^{7/2}-q^{3/2})\left(\frac{[3]}{[2]}\right)^{1/2}\left|\begin{array}{lcl} jk & , & ik \\ k & & j \end{array}\right\rangle\end{aligned}$$

$$\begin{aligned}\check{R}\left|\begin{array}{lcl} ij & , & ii \\ k & & j \end{array}\right\rangle = {} & q^3\left|\begin{array}{lcl} ii & , & ij \\ j & & k \end{array}\right\rangle + (q^{7/2}-q^{3/2})[2]^{1/2}\left|\begin{array}{lcl} ij & , & ii \\ j & & l \end{array}\right\rangle \\ & +(q^{7/2}-q^{3/2})[2]^{-1/2}\left|\begin{array}{lcl} ii & , & ij \\ k & & j \end{array}\right\rangle \\ & +(1-2q^2+q^6)[2]^{-1}\left|\begin{array}{lcl} ij & , & ii \\ k & & j \end{array}\right\rangle + (1-q^2)\frac{[3]^{1/2}}{[2]}\left|\begin{array}{lcl} ik & , & ii \\ j & & j \end{array}\right\rangle\end{aligned}$$

$$\begin{aligned}\check{R}\left|\begin{array}{lcl} jk & , & ij \\ k & & k \end{array}\right\rangle = {} & q^3\left|\begin{array}{lcl} ij & , & jk \\ k & & k \end{array}\right\rangle + (q^{7/2}-q^{3/2})[2]^{1/2}\left|\begin{array}{lcl} ij & , & ik \\ k & & k \end{array}\right\rangle \\ & +(q^{7/2}-q^{3/2})[2]^{-1/2}\left|\begin{array}{lcl} ik & , & jj \\ k & & k \end{array}\right\rangle + (1-2q^2 \\ & +q^6)[2]^{-1}\left|\begin{array}{lcl} jk & , & ij \\ k & & k \end{array}\right\rangle + (1-q^2)\frac{[3]^{1/2}}{[2]}\left|\begin{array}{lcl} jk & , & ik \\ k & & j \end{array}\right\rangle\end{aligned}$$

$$\begin{aligned}\check{R}\left|\begin{array}{lcl} ik & , & ii \\ j & & j \end{array}\right\rangle = {} & q^3\left|\begin{array}{lcl} ii & , & ik \\ j & & j \end{array}\right\rangle + (q^{7/2}-q^{3/2})\left(\frac{[3]}{[2]}\right)^{1/2}\left|\begin{array}{lcl} ii & , & ij \\ k & & j \end{array}\right\rangle \\ & +(1-q^2)\frac{[3]^{1/2}}{[2]}\left|\begin{array}{lcl} ij & , & ii \\ k & & j \end{array}\right\rangle + (q^6-1)[2]^{-1}\left|\begin{array}{lcl} ik & , & ii \\ j & & j \end{array}\right\rangle\end{aligned}$$

$$\begin{aligned}\check{R}\left|\begin{array}{lcl} jk & , & ik \\ k & & j \end{array}\right\rangle = {} & q^3\left|\begin{array}{lcl} ik & , & jk \\ j & & k \end{array}\right\rangle + (q^{7/2}-q^{3/2})\left(\frac{[3]^{1/2}}{[2]}\right)^{1/2}\left|\begin{array}{lcl} ik & , & jj \\ k & & k \end{array}\right\rangle \\ & +(1-q^2)\frac{[3]^{1/2}}{[2]}\left|\begin{array}{lcl} jk & , & ij \\ k & & k \end{array}\right\rangle + (q^6-1)[2]^{-1}\left|\begin{array}{lcl} jk & , & ik \\ k & & j \end{array}\right\rangle\end{aligned}$$

$$\begin{aligned}\check{R}\left|\begin{array}{lcl} ik & , & ij \\ j & & j \end{array}\right\rangle = {} & q^3\left|\begin{array}{lcl} ij & , & ik \\ j & & j \end{array}\right\rangle + (q^4-q^2)\frac{[3]^{1/2}}{[2]}\left|\begin{array}{lcl} ij & , & ij \\ k & & j \end{array}\right\rangle \\ & +(q^6-1)[2]^{-1}\left|\begin{array}{lcl} ik & , & ij \\ j & & j \end{array}\right\rangle + (q^{1/2}-q^{5/2})\left(\frac{[3]}{[2]}\right)^{1/2}\left|\begin{array}{lcl} jj & , & ii \\ k & & j \end{array}\right\rangle\end{aligned}$$

$$\check{R}\left|\begin{array}{lcl} ik & , & ik \\ k & & j \end{array}\right\rangle = q^3\left|\begin{array}{lcl} ik & , & ik \\ j & & k \end{array}\right\rangle + (q^4-q^2)\frac{[3]^{1/2}}{[2]}\left|\begin{array}{lcl} ik & , & ij \\ k & & k \end{array}\right\rangle$$

$$+(q^6-1)[2]^{-1}\left|\begin{matrix} ik & , & ik \\ k & & j \end{matrix}\right\rangle+(q^{1/2}-q^{5/2})\left(\frac{[3]}{[2]}\right)^{1/2}\left|\begin{matrix} jk & , & ii \\ k & & k \end{matrix}\right\rangle$$

$$\begin{aligned}\check{R}\left|\begin{matrix} jj & , & ii \\ k & & j \end{matrix}\right\rangle &= q^2\left|\begin{matrix} ii & , & jj \\ j & & k \end{matrix}\right\rangle+(q^{7/2}-q^{3/2})[2]^{1/2}\left|\begin{matrix} ij & , & ij \\ j & & k \end{matrix}\right\rangle \\ &\quad +(q^{9/2}-q^{5/2})[2]^{-1/2}\left|\begin{matrix} ij & , & ij \\ k & & j \end{matrix}\right\rangle+(q^{1/2}-q^{5/2})\left(\frac{[3]}{[2]}\right)^{1/2}\left|\begin{matrix} ik & , & ij \\ j & & j \end{matrix}\right\rangle \\ &\quad +(q-2q^3+q^5)\left|\begin{matrix} jj & , & ii \\ k & & j \end{matrix}\right\rangle\end{aligned}$$

$$\begin{aligned}\check{R}\left|\begin{matrix} jk & , & ii \\ k & & k \end{matrix}\right\rangle &= q^2\left|\begin{matrix} ii & , & jk \\ k & & k \end{matrix}\right\rangle+(q^{7/2}-q^{3/2})[2]^{1/2}\left|\begin{matrix} ij & , & ik \\ k & & k \end{matrix}\right\rangle \\ &\quad +(q^{9/2}-q^{5/2})[2]^{-1/2}\left|\begin{matrix} ik & , & ij \\ k & & k \end{matrix}\right\rangle \\ &\quad +(q^{1/2}-q^{5/2})\left(\frac{[3]}{[2]}\right)^{1/2}\left|\begin{matrix} ik & , & ik \\ k & & j \end{matrix}\right\rangle+(q-2q^3+q^5)\left|\begin{matrix} jk & , & ii \\ k & & k \end{matrix}\right\rangle\end{aligned}$$

$$\check{R}\left|\begin{matrix} ij & , & ii \\ k & & k \end{matrix}\right\rangle = q^3\left|\begin{matrix} ii & , & ij \\ k & & k \end{matrix}\right\rangle+(q^5-q)\left|\begin{matrix} ij & , & ii \\ k & & k \end{matrix}\right\rangle+(q^{-1/2}-q^{3/2})[2]^{-1/2}\left|\begin{matrix} ik & , & ii \\ k & & j \end{matrix}\right\rangle$$

$$\begin{aligned}\check{R}\left|\begin{matrix} jj & , & ij \\ k & & k \end{matrix}\right\rangle &= q^3\left|\begin{matrix} ij & , & jj \\ k & & k \end{matrix}\right\rangle+(q^5-q)\left|\begin{matrix} jj & , & ij \\ k & & k \end{matrix}\right\rangle \\ &\quad +(q^{-1/2}-q^{3/2})[2]^{-1/2}\left|\begin{matrix} jk & , & ij \\ k & & j \end{matrix}\right\rangle\end{aligned}$$

$$\check{R}\left|\begin{matrix} ik & , & ii \\ j & & k \end{matrix}\right\rangle = q^3\left|\begin{matrix} ii & , & ik \\ k & & j \end{matrix}\right\rangle+(q^{7/2}-q^{3/2})\left(\frac{[3]}{[2]}\right)^{-1/2}\left|\begin{matrix} ik & , & ii \\ k & & j \end{matrix}\right\rangle$$

$$\check{R}\left|\begin{matrix} jj & , & ik \\ k & & j \end{matrix}\right\rangle = q^3\left|\begin{matrix} ik & , & jj \\ j & & k \end{matrix}\right\rangle+(q^{7/2}-q^{3/2})\left(\frac{[3]}{[2]}\right)^{-1/2}\left|\begin{matrix} jk & , & ij \\ k & & j \end{matrix}\right\rangle$$

$$\begin{aligned}\check{R}\left|\begin{matrix} ik & , & ii \\ k & & j \end{matrix}\right\rangle &= q^2\left|\begin{matrix} ii & , & ik \\ j & & k \end{matrix}\right\rangle+(q^{7/2}-q^{3/2})[2]^{-1/2}\left|\begin{matrix} ii & , & ij \\ k & & k \end{matrix}\right\rangle \\ &\quad +(q^{-1/2}-q^{3/2})[2]^{-1/2}\left|\begin{matrix} ij & , & ii \\ k & & k \end{matrix}\right\rangle+(q^{7/2}-q^{3/2})\left(\frac{[3]}{[2]}\right)^{-1/2}\left|\begin{matrix} ii & , & ik \\ k & & j \end{matrix}\right\rangle \\ &\quad +(q^{7/2}-q^{3/2})\left(\frac{[3]}{[2]}\right)^{-1/2}\left|\begin{matrix} ik & , & ik \\ j & & k \end{matrix}\right\rangle+(q-2q^3+q^5)\left|\begin{matrix} ik & , & ii \\ k & & j \end{matrix}\right\rangle\end{aligned}$$

$$\begin{aligned}\check{R}\left|\begin{matrix} jk & , & ij \\ k & & j \end{matrix}\right\rangle &= q^2\left|\begin{matrix} ij & , & jk \\ j & & k \end{matrix}\right\rangle+(q^{7/2}-q^{3/2})[2]^{-1/2}\left|\begin{matrix} ij & , & jj \\ k & & k \end{matrix}\right\rangle \\ &\quad +(q^{-1/2}-q^{3/2})[2]^{-1/2}\left|\begin{matrix} jj & , & ij \\ k & & k \end{matrix}\right\rangle+(q^{7/2}-q^{3/2})\left(\frac{[3]}{[2]}\right)^{-1/2}\left|\begin{matrix} ik & , & jj \\ j & & k \end{matrix}\right\rangle \\ &\quad +(q^{7/2}-q^{-3/2})\left(\frac{[3]}{[2]}\right)^{-1/2}\left|\begin{matrix} jj & , & ik \\ k & & j \end{matrix}\right\rangle \\ &\quad +(q-2q^3+q^5)\left|\begin{matrix} jk & , & ij \\ k & & j \end{matrix}\right\rangle\check{R}\left|\begin{matrix} ij & , & ij \\ k & & k \end{matrix}\right\rangle \\ &= q^3\left|\begin{matrix} ij & , & ij \\ k & & k \end{matrix}\right\rangle+(q^{-2}-2q^2+q^4)[2]^{-1}\left|\begin{matrix} jk & , & ii \\ k & & j \end{matrix}\right\rangle \\ &\quad +(q^4-1)\left|\begin{matrix} jj & , & ii \\ k & & k \end{matrix}\right\rangle+(q^3-q)[2]^{-1}\left|\begin{matrix} ik & , & ij \\ k & & j \end{matrix}\right\rangle\end{aligned}$$

$$
\begin{aligned}
\check{R}\left|\begin{array}{lcl} jk & , & ii \\ k & & j \end{array}\right\rangle &= q\left|\begin{array}{lcl} ii & , & jk \\ j & & k \end{array}\right\rangle + (q^2-1)\left|\begin{array}{lcl} ij & , & ik \\ j & & k \end{array}\right\rangle \\
&\quad + (q^{-2}-2q^2+q^4)[2]^{-1}\left|\begin{array}{lcl} ij & , & ij \\ k & & k \end{array}\right\rangle \\
&\quad + (q^4-q^2)\frac{[3]^{1/2}}{[2]}\left|\begin{array}{lcl} ij & , & ik \\ k & & j \end{array}\right\rangle + (q^4-q^2)\frac{[3]^{1/2}}{[2]}\left|\begin{array}{lcl} ik & , & ij \\ j & & k \end{array}\right\rangle \\
&\quad - (q^4-q^{-2})[2]^{-1}\left|\begin{array}{lcl} ik & , & ik \\ j & & j \end{array}\right\rangle - [2]^2\left(\left|\begin{array}{lcl} jj & , & ii \\ k & & k \end{array}\right\rangle + \left|\begin{array}{lcl} ik & , & ij \\ k & & j \end{array}\right\rangle\right) \\
&\quad + (q^{-3}-2q^{-1}+2q-2q^3+q^5)\left|\begin{array}{lcl} jk & , & ij \\ k & & j \end{array}\right\rangle \\
\check{R}\left|\begin{array}{lcl} ik & , & ij \\ j & & k \end{array}\right\rangle &= q^3\left|\begin{array}{lcl} ij & , & ik \\ k & & j \end{array}\right\rangle + (q^3-q)\frac{[3]^{1/2}}{[2]}\left|\begin{array}{lcl} ik & , & ij \\ k & & j \end{array}\right\rangle \\
&\quad + (q^4-q^2)\frac{[3]^{1/2}}{[2]}\left|\begin{array}{lcl} jk & , & ii \\ k & & j \end{array}\right\rangle \\
\check{R}\left|\begin{array}{lcl} ij & , & ik \\ k & & j \end{array}\right\rangle &= q^3\left|\begin{array}{lcl} ik & , & ij \\ j & & k \end{array}\right\rangle + (q^3-q)\frac{[3]^{1/2}}{[2]}\left|\begin{array}{lcl} ik & , & ij \\ k & & j \end{array}\right\rangle \\
&\quad + (q^4-q^2)\frac{[3]^{1/2}}{[2]}\left|\begin{array}{lcl} jk & , & ii \\ k & & j \end{array}\right\rangle \\
\check{R}\left|\begin{array}{lcl} ik & , & ik \\ j & & j \end{array}\right\rangle &= q^3\left|\begin{array}{lcl} ik & , & ik \\ j & & j \end{array}\right\rangle + (q^5-q^{-1})[2]^{-1/2}\left|\begin{array}{lcl} ik & , & ij \\ k & & j \end{array}\right\rangle \\
&\quad - (q^4-q^{-2})[2]^{-1/2}\left|\begin{array}{lcl} jk & , & ii \\ k & & j \end{array}\right\rangle \\
\check{R}\left|\begin{array}{lcl} ik & , & ij \\ k & & j \end{array}\right\rangle &= q\left|\begin{array}{lcl} ij & , & ik \\ j & & k \end{array}\right\rangle + (q^3-q)\frac{[3]^{1/2}}{[2]}\left(\left|\begin{array}{lcl} ij & , & ik \\ k & & j \end{array}\right\rangle + \left|\begin{array}{lcl} ik & , & ij \\ j & & k \end{array}\right\rangle\right) \\
&\quad + (q^5-q^{-1})[2]^{-1}\left|\begin{array}{lcl} ik & , & ik \\ j & & j \end{array}\right\rangle + (q^{-1}-q-q^3+q^5)\left|\begin{array}{lcl} ik & , & ij \\ k & & j \end{array}\right\rangle \\
&\quad + (q^{-1}-q)\left|\begin{array}{lcl} jj & , & ii \\ k & & k \end{array}\right\rangle - [2]^2\left|\begin{array}{lcl} jk & , & ii \\ k & & j \end{array}\right\rangle \\
\check{R}\left|\begin{array}{lcl} jj & , & ii \\ k & & k \end{array}\right\rangle &= q\left|\begin{array}{lcl} ii & , & jj \\ k & & k \end{array}\right\rangle + (q^4-1)\left|\begin{array}{lcl} ij & , & ij \\ k & & k \end{array}\right\rangle + (q^{-1}-q)\left|\begin{array}{lcl} ik & , & ij \\ k & & j \end{array}\right\rangle \\
&\quad + (q^{-1}-q-q^3+q^5)\left|\begin{array}{lcl} jj & , & ii \\ k & & k \end{array}\right\rangle - [2]^2\left|\begin{array}{lcl} jk & , & ii \\ k & & j \end{array}\right\rangle
\end{aligned}
\qquad (42)
$$

where we always assume that $1 \leqslant i < j < k < l \leqslant n$. The above results exhaust all the $\check{R}$-matrix elements for $[1]\times[1]$, $[2]\times[2]$, and $[1^2]\times[1^2]$. Because there are too many matrix elements for $[21]\times[21]$, we only list those for $U_q(3)$ completely. But these $U_q(3)\,\check{R}$-matrix elements are also those of $U_q(n)$; i.e. the results are n independent. Secondly, another advantage of this method is that it leaves the coproduct multiplicity problem untouched in contrast with the projection operator technique, in which one should derive the CG matrix of the corresponding quantum groups. Sometimes the coproduct concerned is not multiplicity free. For example, in our $[21]\times[21]$ case, the resultant irrep [21] is of multiplicity two. However, in our method the $\check{R}$ operator is acting on the uncoupled basis vectors of the corresponding quantum groups, and has nothing to do with the CG coefficients of $[21]\times[21]$. From these simple examples, it can easily be seen that the $\check{R}$-matrix elements obtained in this way are n independent; i.e. all the $\check{R}$-matrix elements of $U_q(n)$ listed apply for arbitrary n.

The procedures for evaluating $\check{R}$-matrix elements of other quantum groups are very similar. For example, we can evaluate $\check{R}$-matrix elements for the G_2 case with the help of Kalfagianni algebra [22]. Hence the problem for constructing the $\check{R}$-matrix is switched to find out all the possible algebraic realizations of the braid group. It should be noted that this method will also become very tedious when the rank of the irrep increases. In such a case, the $\check{R}$ operator is expressed in terms of a lengthy braid-group generator product. Though this method is not simpler than the other methods for higher-dimensional irreps of the quantum group, it shows us a new way to calculate $\check{R}$-matrix elements, a new perspective of $\check{R}$ matrices, and a transparent view of its braid-group structure.

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